Mathematical Model of the Approximate Reasoning in FLC Systems with Uninorm-residuum

Marta Takács
Department of Information Technology
Budapest Polytechnic
H-1081 Budapest, Népszínház utca 8.
Hungary
takacs.marta@nik.bmf.hu

Abstract –**The paper introduces novel reasoning systems in a uninorm environment. Based on the definitions and theorems for lattice ordered monoids and left continuous uninorms and t-norms, certain distance-based operators are focused on, with the help of which the uninorm-residuum based approximate reasoning system becomes possible in Fuzzy Logic Control (FLC) systems.**

I. INTRODUCTION

The modelling of the uncertain processes in our modern society is a very complex problem. Since the systems are multi-criterial and multipart, decision processes become increasingly vague and hard to analyze. The human brain possesses some special characteristics that enable it to learn and reason in a vague and fuzzy environment. Naturally, there have been models that have investigated the behaviour of complex systems before the introduction of the fuzzy systems, but fuzzy systems have proven to be to a greater or lesser degree more useful than the classical models.

The possibility theory is, for example, based on experimental observations and analysis of the statistical data. With respect to this method the fuzzy decision making model needs nothing but the experts' observations to construct the fuzzy rule base, and since then the fuzzy system works on line for all real time inputs.

The real time functioning of the dynamic engineering system is a necessary condition. The earlier, differential-equation based models, with the expansion of the complexity of the real systems are growing out of all proportion, and it is very hard to construct a real time decision making model under those circumstances. Since the fuzzy logic control systems are based on linguistic variables [14], and fuzzy approximate reasoning, and therefore they are successful and effective [15].

Generally, the fundamental of the decision making in fuzzy based real systems is the approximate reasoning, which is a rule-based system. Knowledge representation in a rule-based system is done by means of IF…THEN rules. Furthermore, approximate reasoning systems allow fuzzy inputs, fuzzy antecedents, fuzzy consequents. "Informally, by approximate or, equivalently, fuzzy reasoning, we mean the process or processes by which a possibly imprecise conclusion is deduced from a collection of imprecise premises. Such reasoning is, for the most part, qualitative rather than quantitative in nature and almost all of it falls outside of the domain of applicatibility of classical logic", [13]. This fuzzy representation allows a closer match with many of the important concepts of practical affairs, which lack the sharp boundaries assumed by classical logic.

The fundamental point arising from the classical two-valued logic is that it imposes a dichotomy on any mathematical model. However, in many cases of daily life, a borderline between the two possibilities is not evident. There are, of course the generalizations of two-valued logic, the multi-valued logics. In this model there may be a finite or infinite number of truth values, that is, an infinite number of degrees to which a property may be present. But in contrast with multi-valued logics, "fuzzy logic differs from conventional logical systems in that it aims at providing a model for approximate rather than precise reasoning", [1].

Fuzzy logic also provides a system that is sufficiently flexible and expressive to serve as a natural framework for the semantics of natural languages. Also in fuzzy logic truth itself is allowed to be such as "quite true", "more or less true",[2].

Since the fuzzy systems were introduced and developed in the mid-sixties, a lot of applications have been found. They have been used in various areas, such as technical system control, economic trend prognosis, or medicine, to just name the most prominent ones. The latest field of application for the nonlinear control via TP model transformation, with the TORA System example [20].

Based on these it can be stated that fuzzy systems are very much application-oriented. However, these systems need to be integrated into the well-known mathematical theories, and if there are new topics to be introduced, they have to be axiomatized and mathematically defined.

Although application-oriented fuzzy systems seek to be simple and comprehensible, it is obvious that they are heavily related to the fields of classical multi-valued logic, operation research and functional analysis. There are numerous models that are yet to gain exact mathematical description, but have already proven their applicability in practice.

This latter question seems to be answered. Experts are confident in using special types of operations in fuzzy systems, such as t-norms, t-conorms, uninorms, and more generally, aggregation operators, and researchers are more and more meticulous in providing exact mathematical definitions for those. The focus is on certain properties of operators (continuity and representability, for instance), and those classes of operators are highlighted that correspond to the applications. This paper is a step towards the investigation of this problem by reviewing a specific case, where the investigated structure is a real semi-ring with pseudo-operations [16].

The question raised previously was whether there are general operation groups which satisfy the residuum-based approximate reasoning, but at the same time are easily comprehensible and acceptable to application-oriented experts. How come the residuum-based approximate reasoning is not as wide-spread in mathematical logic as the Mamdani-type? Why is the application of operations in fuzzy systems limited to that of the minimum and the product? In [11] only a partial answer to this question is laid out. The axiom system presented in [11] greatly contributes to this. Situated between the theses, these axioms declare the expectations of the approximate reasoning systems, and it becomes clear to what degree they satisfy or violate this system.

As is the case with every research, the completion of each step of those researches has raised further questions, such as the relationship between the reasoning system and the fuzzy integral, or the role of fuzzy measures in approximate reasoning.

The theoretical basis used in [11] provided by [7], which is currently the leading work in the world concerning fuzzy operators. It represents the monography describing t-norms, t-conorms and related operators for fuzzy sets and numbers.

The other basic background of this research are the distance-based operators introduced by Rudas in his work [8],[9]. The characteristics of those operators were further investigated by Rudas and the author of this paper in various joint projects [10].

The research heavily relies on approximate reasoning and fuzzy logic theory (focusing on implications). The following works were mainly used in this investigation: [2], [4].

Concerning the structure of the work is the next: the first section an overview of uninorm operators is given, with its role in residuum-based approximate reasoning. It is emphasized which operators from distance based operator group are the ones corresponding to the uninorm-based residuum known so far. The next section also introduces novel reasoning systems in a uninorm environment. Based on the theorems shown in the first section, certain distance-based operators are focused on, with the help of which the residuum-based approximate reasoning system becomes possible.

II UNINORMS

Both the neutral element 1 of a t-norm and the neutral element 0 of a t-conorm are boundary points of the unit interval. However, there are many important operations whose neutral element is an interior point of the underlying set. The fact that the first three axioms (commutativity, associativity, montonocity) coincide for t-norms and for t-conorms, i.e., the only axiomatic difference lies in the location of the neutral element, has led to the introduction of a new class of binary operations closely related to t-norms and t-conorms.

A *uninorm* is a binary operation U on the unit interval, i.e., a function $U:[0,1]^2 \to [0,1]$ which satisfies the following properties for all $x,y,z \in [0,1]$

(U1) $U(x,y)=U(y,x)$, i.e. the uninorm is commutative,

(U2) $U(U(x,y),z)=U(x,U(y,z))$, i.e. the uninorm is associative,

(U3) $x \le y \Rightarrow U(x,z) \le U(y,z)$, i.e. the uninorm monotone,

(U4) $U(e,x)=x$, i.e., a neutral element exists, which is $e \in [0,1]$.

On the one hand the practical motivations for the introduction of uninorms were the applications from multicriteria decision making, where the aggregation is one of the key issues. Some alternatives are evaluated from several points of view. Each evaluation is a number from the unit interval. Let the level of satisfaction be $e \in \left]0,1\right[$. If all criteria are satisfied to at least e-extent then we would like to assign a high aggregated value to this alternative. The opposite of that is if all evaluations are below e then we would like to assign a low aggregated value to this alternative. But if there are evaluations below and above e, an aggregated value ought to be assigned somewhere in between. Such situations can be modelled by uninorms, leading to the particular classes introduced by [12].

2.1. Lattice ordered monoids and left continuous uninorms and t-norms

Let L be a non-empty set. Lattice is a partially (totally) ordered set which for any two elements $x,y \in L$ also contains their *join* $x \vee y$ (i.e., the least upper bound of the set $\{x,y\}$), and their *meet* $x \wedge y$ (i.e., the greatest lower bound of the set $\{x,y\}$), denoted by $(L,\preceq)$. Secondly, $(L,*)$ is a semi-group with the neutral element. Following [16], [17] let the following be introduced:

Definition 2.1.

Let $(L,\preceq)$ be a lattice and $(L,*)$ a semi-group with the neutral element.

(i) The triple $(L,*,\preceq)$ is called a *lattice-ordered monoid* (or an *l-monoid*) if for all $x,y,z \in L$ we have

(LM1) $x*(y \vee z)=(x*y)\vee(x*z)$ and

(LM2) $(x \vee y)*z=(x*z)\vee(y*z)$.

(ii) An $(L,*,\preceq)$ *l*- monoid is said to be *commutative* if the semi-group $(L,*)$ is commutative.

(iii) A commutative $(L,*,\preceq)$ *l*- monoid is said to be *commutative, residuated l-monoid* if there exists a further binary operation $\to_*$ on L, i.e., a function $\to_*: L^2 \to L$ (*the $*$ residuum*), such that for all $x,y,z \in L$ we have

(*Res*) $x * y \preceq z$ if and only if $x \preceq (y \to_* z)$.

(iv) An *l*-monoid $(L,*,\preceq)$ is called an *integral* if there is a greatest element in the lattice $(L,\preceq)$ (often called the universal upper bound) which coincides with the neutral element of the semi-group $(L,*)$.

Obviously, each *l*-monoid $(L,*,\preceq)$ is a partially ordered semi-group, and in the case of commutativity the axioms (*LM1*) and (*LM2*) are equivalent.

In the following investigations the focus will be on the lattice $([0,1],\leq)$, we will usually work with a complete lattice, i.e., for each subset *A* of *L* its join $\vee A$ and its $\wedge A$ exist and are contained in *L*. In this case, *L* always has a greatest element, also called the *universal upper bound*.

***Example 2.1.*.** If we define $*:[0,1]^2 \to [0,1]$ by

$$x * y = \begin{cases} min(x,y) & if \quad x+y \leq 1 \\ max(x,y) & otherwise \end{cases}$$

then $([0,1],*,\leq)$ is a commutative, residuated *l*-monoid, and the *-residuum is given by

$$x \to_* y = \begin{cases} max(1-x,y) & if \quad x \leq y \\ min(1-x,y) & otherwise \end{cases}.$$

It is not an integral, since the neutral element is 0.5.

The operation * results in an *uninorm*, and special types of distance based operators (see [11], *Section 2.* and [3].

The following result is on important characterization of left-continuous uninorms.

Theorem 2.1.

For each function $U:[0,1]^2 \to [0,1]$ the following are equivalent:

(*i*) $([0,1],U,\leq)$ is a commutative, residuated *l*-monoid, with a neutral element

(*ii*) *U* is a left continuous uninorm.

In this case the *U*-residuum $\to_U$ is given by

(*ResU*)

$$x \to_U y = sup\{z \in [0,1] | U(x,z) \leq y\}.$$

Proof. (In the [11])

The work of De Baets, B. and Fodor, J. [3] presents general theoretical results related to residual implicators of uninorms, based on residual implicators of t-norms and t-conorms.

Residual operator R_U, considering the uninorm U, can be represented in the following form:

$$R_U(x,y) = sup\{z | z \in [0,1] \wedge U(x,z) \leq y\}.$$

Uninorms with the neutral elements $e = 0$ and $e = 1$ are t-norms and t-conorms, respectively, and related residual operators are widely discussed, we also find suitable definitions for uninorms with neutral elements $e \in]0,1[$.

If we consider a uninorm U with the neutral element $e \in]0,1[$, then the binary operator R_U is an implicator if and only if $(\forall z \in]e,1[)(U(0,z)=0)$. Furthermore R_U is an implicator if *U* is a disjunctive right-continuous idempotent uninorm with unary operator g satisfying $(\forall z \in [0,1])(g(z)=0 \Leftrightarrow z=1)$.

The residual implicator R_U of uninorm *U* can be denoted by Imp_U.

Consider a uninorm *U*, then R_U is an implicator in the following cases:

(i) *U* is a conjunctive uninorm,

(ii) *U* is a disjunctive representable uninorm,

(iii) *U* is a disjunctive right-continuous idempotent uninorm with unary operator g satisfying

$(\forall z \in [0,1])(g(z)=0 \Leftrightarrow z=1)$.

Theorem 2.1. implies in a special case Proposition 2.47. from [7]:

Corollary 2.1.

For each function $T:[0,1]^2 \to [0,1]$ the following are equivalent:

(*i*) $([0,1],T,\leq)$ is a commutative, residuated integral *l*-monoid,

(*ii*) *T* is a left continuous t-norm.

In this case the *T*-residuum $\to_T$ is given by (*ResT*)

$$x \to_T y = sup\{z \in [0,1] | T(x,z) \leq y\}.$$

Because of its interpretation in [0,1]-valued logics, the *T*-residuum $\to_T$ is also called *residual implication* (or briefly *R-implication*). [6]

III. APPROXIMATE REASONING AND THE FUZZY LOGIC CONTROL

In control theory and also in theory of the approximate reasoning introduced by Zadeh in 1979, [14] much of the knowledge of system behaviour and system control can be stated in the form of if-then rules. The Fuzzy Logic Control, FLC has been carried out searching for different mathematical models in order to supply these rules.

In most sources it was suggested to represent an

if *x* is *A* then *x* is *B*

rule in the form of fuzzy implication (shortly *Imp*(*A*,*B*)), relation (shortly *R*(*A*,*B*)), or simply as a connection (for example as a t-norm, *T*(*A*,*B*)) between the so called rule premise: *x* is *A* and rule consequence: *y* is *B*. Let *x* be from universe *X*, y from universe *Y*, and let *x* and *y* be linguistic variables. Fuzzy set *A* in *X* is characterised by its membership function μ_A: $x \to [0,1]$. The most significant differences between the models of FLC-s lie in the definition of this connection, relation or implication.

The other important part of the FLC is the inference mechanism. One of the widely used methods is the Generalised Modus Ponens (GMP), in which the main point is, that the inference y is B' is obtained when the propositions are:

- the i^{th} rule from the rule system of n rules: if x is A_i then y is B_i
- and the system input x is A'.

GMP sees the real influences of the implication or connection choice on the inference mechanisms in fuzzy systems ([4], [13]). Usually the general rule consequence for i^{th} rule from a rule system is obtained by

$$B_i'(y) = \sup_{x \in X} T(A'(x), Imp(A_i(x), B_i(y))).$$

3.1. Residuum-based approximate reasoning with distance-based uninorms

In fact the uninorms offer new possibilities in fuzzy approximate reasoning, because the low level of covering over of rule premise and rule input has measurable influence on rule output as well. In some applications the meaning of that novel t-norms, has practical importance. The modified Mamdani's approach, with similarity measures between rule premises and rule input, does not rely on the compositional rule inference any more, but still satisfies the basic conditions supposed for the approximate reasoning for a fuzzy rule base system [19].

Having results from [3], we can introduce residuum-based inference mechanism using distance-based uninorms.

3.2. Modified distance-based operators

The distance-based operators can be expressed by means of the min and max operators as follows (the only modification on distance based operators described in [9] is the boundary condition for neutral element e):

the *maximum distance minimum operator with respect* to $e \in]0,1]$ is defined as

$$max_e^{min} = \begin{cases} max(x,y), & \text{if } y > 2e - x \\ min(x,y), & \text{if } y < 2e - x, \\ min(x,y), & \text{if } y = 2e\text{-}x \end{cases}$$

the *minimum distance minimum operator* with respect to $e \in [0,1[$ is defined as

$$min_e^{min} = \begin{cases} min(x,y), & \text{if } y > 2e - x \\ max(x,y), & \text{if } y < 2e - x, \\ min(x,y), & \text{if } y = 2e\text{-}x \end{cases}$$

the *maximum distance maximum operator with respect to* $e \in]0,1]$ is defined as

$$max_e^{max} = \begin{cases} max(x,y), & \text{if } y > 2e - x \\ min(x,y), & \text{if } y < 2e - x, \\ max(x,y), & \text{if } y = 2e\text{-}x \end{cases}$$

the *minimum distance maximum operator with respect to* $e \in [0,1[$ is defined as

$$min_e^{max} = \begin{cases} min(x,y), & \text{if } y > 2e - x \\ max(x,y), & \text{if } y < 2e - x. \\ max(x,y), & \text{if } y = 2e\text{-}x \end{cases}$$

The distance-based operators have the following properties

max_e^{min} *and* max_e^{max} are uninorms,

the dual operator of the uninorm max_e^{min} *is* max_{1-e}^{max}, *and*

the dual operator of the uninorm max_e^{max} *is* max_{1-e}^{min}.

Based on results from [3] and [4] we conclude:

Operator $max_{0.5}^{min}$ is a conjunctive left-continuous idempotent uninorm with neutral element $e \in]0,1]$ with the super-involutive decreasing unary operator $g(x) = 2e - x = 2 \cdot 0.5 - x \Rightarrow g(x) = 1 - x$.

Operator $min_{0.5}^{max}$ is a disjunctive right-continuous idempotent uninorm with neutral element $e \in]0,1]$ with the sub-involutive decreasing unary operator $g(x) = 2e - x = 2 \cdot 0.5 - x \Rightarrow g(x) = 1 - x$.

3.3. Idempotent uninorms and the residual implicators of uninorms

A binary operator V is called idempotent, if $V(x,x) = x, (\forall x \in X)$. It is well known, that the only idempotent t-norm is *min*, and the only t-conorm is *max*.

In [3], by De Baets, B. and Fodor, J., has studied two important classes of uninorms: the class of left-continuous and the class of right-continuous ones.

If we suppose a unary operator g on set [0,1], then g is called

(*i*) sub-involutive if $g(g(x)) \le x$ for $(\forall x \in [0,1])$, and

(*ii*) super-involutive if $g(g(x)) \ge x$ for $(\forall x \in [0,1])$.

A binary operator U is a conjunctive left-continuous idempotent uninorm with neutral element $e \in]0,1]$ if and only if there exist a super-involutive decreasing unary operator g with fixpoint e and $g(0) = 1$ such that U for any $\forall (x,y) \in [0,1]^2$ is given by

$$U(x,y) = \begin{cases} min(x,y) & if \quad y \le g(x) \\ max(x,y) & elsewhere \end{cases}.$$

A binary operator U is a dijunctive right-continuous idempotent uninorm with neutral element $e \in [0,1[$ if and only if there exist a sub-involutive decraesing unary operator g with fixpoint e and $g(1) = 0$ such that U for any $\forall (x,y) \in [0,1]^2$ is given by

$$U(x,y) = \begin{cases} max(x,y) & if \quad y \ge g(x) \\ min(x,y) & elsewhere \end{cases}.$$

Residual operator R_U, considering the uninorm U, can be represented in the following form:

$$R_U(x,y)=sup\{z \mid z\in[0,1] \wedge U(x,z)\le y\}.$$

Uninorms with neutral elements $e=0$ and $e=1$ are t-norms and t-conorms, respectively, and related residual operators are widely discussed.

If we consider a uninorm U with neutral element $e\in]0,1[$, then the binary operator R_U is an implicator if and only if $(\forall z\in]e,1[)(U(0,z)=0)$. Furthermore R_U is an implicator if U is a disjunctive right-continuous idempotent uninorm with unary operator g satisfying $(\forall z\in[0,1])(g(z)=0 \Leftrightarrow z=1)$.

The residual implicator R_U of uninorm U can be denoted by Imp_U.

3.4. Residual implicators of distance based operators

According to Theorem 8. in [3] we introduce implicator of distance based operator $max_{0.5}^{min}$.

Consider the conjunctive left-continuous idempotent uninorm $max_{0.5}^{min}$ with the unary operator $g(x)=1-x$, then its residual implicator $Imp_{max_{0.5}^{min}}$ is given by

$$Imp_{max_{0.5}^{min}}=\begin{cases} max(1-x,y) & if \quad x\le y \\ min(1-x,y) & elsewhere \end{cases} \qquad (3.1.).$$

3.5. Residuum-based approximate reasoning with distance based operator

Although the minimum plays an exceptional role in fuzzy control theory, there are situations requiring new models. In system control one would intuitively expect: to make the powerful coincidence between fuzzy sets stronger, and the weak coincidence even weaker. The distance-based operators group satisfy these properties. Let we consider a mathematical approach: residuum-based approximate reasoning and inference mechanism. Hence, and because of the results from sections of this paper we can n consider the general rule consequence for i-th rule from a rule system as

$$B_i'(y)=\sup_{x\in X}\left(max_{0.5}^{min}\left(A'(x),Imp_{max_{0.5}^{min}}(A_i(x),B_i(y))\right)\right),$$

or, using formula (*3.1.*)

$$B_i'(y)=\sup_{x\in X}\begin{cases} max_{0.5}^{min}(A'(x),max(1-A_i(x),B_i(y))) & if\ A_i(x)\le B_i(y) \\ max_{0.5}^{min}(A'(x),min(1-A_i(x),B_i(y))) & elsewhere \end{cases}$$

The rule base output is constructed as a crisp value calculated with a defuzzification model, from rule base output. Rule base output is an aggregation of all rule consequences $B_i'(y)$, from the rule base. As aggregation operator, in this case, dual operator $max_{0.5}^{max}$ of $max_{0.5}^{min}$ can be used.

$$B'_{out}(y)=max_{0.5}^{max}(B_n'(y),max_{0.5}^{max}(B_{n-1}'(y),max_{0.5}^{max}(....,max_{0.5}^{max}(B_2'(y),B_1'(y))))).$$

Taken into account Proposition 13. from [3], it can be conclude, that conjunctive left-continuous idempotent uninorm $max_{0.5}^{min}$ and its implicator $Imp_{max_{0.5}^{min}}$ satisfy the inequality

$$B'_i(y)=max_{0.5}^{min}\left(A'(x),Imp_{max_{0.5}^{min}}(A_i(x),B_i(y))\right)\le B_i(y)$$

for i-th rule in rule base system, if $A'(x)=A_i(x)$ for all $x\in X$. It means, that this type of reasoning partially satisfies the conditions for approximate reasoning, hence $B'_i(y)=B_i(y)$ or $B'_i(y)<B_i(y)$ if $A'(x)=A_i(x)$ for all $x\in X$.

IV. CONCLUSION

Based on the definitions and theorems for lattice ordered monoids and left continuous uninorms and t-norms, certain distance-based operators are focused on, with the help of which the uninorm-residuum based approximate reasoning system becomes possible in Fuzzy Logic Control (FLC) systems. If we consider the conjunctive left-continuous idempotent uninorm $max_{0.5}^{min}$ with the unary operator $g(x)=1-x$, then we can construct its residual implicator $Imp_{max_{0.5}^{min}}$, and integrate it in the decision making system of the FLC.

Acknowledgement

The research was partially supported by the Hungarian Scientific Research Project OTKA T037304.

V. REFERENCES

[1] Bellmann, R.E., Zadeh., L.A., (1977), *Local and fuzzy logic,* Modern Uses of Multiple-Valued Logic, edited by Dunn, J.M., Epstein, G., Reidel Publ., Dordrecht, The Netherlands, pp. 103-165.

[2] Da, R., (1990), *A critical study of widely used fuzzy implication operators and their influence on the inference rules in fuzzy expert systems,* State Univ. of Ghent, Faculty of Science,1990.

[3] De Baets, B., Fodor, J., (1999), *Residual operators of uninorms,*, Soft Computing 3., (1999), pp. 89-100.

[4] Fodor, J., Rubens, M., (1994), *Fuzzy Preference Modeling and Multi-criteria Decision Support.* Kluwer Academic Pub.,1994.

[6] Fodor, J., (1996), *Fuzzy Implications,* Proc. Of International Panel Conference on Soft and Intelligent Computing, Budapest, ISBN 963 420 510 0, pp. 91-98.

[7] Klement, E.P., Mesiar, R, Pap, E., *'Triangular Norms'*, Kluwer Academic Publishers, 2000, ISBN 0-7923-6416-3

[8] Rudas, I., *Absorbing-Norms,* in Proceedings of the 3th International Symposium of Hungarian Researchers Computational Intelligence, Buadpest, Nov. 2002., pp. 25-43.

[9] Rudas, Imre; Kaynak, O., *New Types of Generalized Operations,* Computational Intelligence, Soft Computing and Fuzzy-Neuro Integration with Applications, Springer NATO ASI Series. Series F, Computer and Systems Sciences, Vol. 192. 1998. (O. Kaynak, L. A. Zadeh, B. Turksen,I. .J. Rudas editors), pp. 128-156.

[10]Takacs, M., Rudas I. J., (1999), *Generalized Mamdani Type Fuzzy Controllers,* in Proceedings of Eurofuse-SIC 99, Budapest, May, 1999. pp. 162-165.

[11]Takacs, M., Approximate Reasoning in Fuzzy Systems Based On Pseudo-Analysis, Phd Thesis, Univ. of Novi Sad, 2004.

[12] Yager, R. R., Rybalov, A., (1996), *Uninorm aggregation operators,* Fuzzy sets and systems 80., pp. 111-120.

[13] Zadeh, L. A., *A Theory of approximate reasoning,* In Hayes, J., and editors, Mashine Intelligence, Vol. 9, Halstead Press, New York, 1979., pp. 149-194.

[14]Zadeh, L. A., (1999), *From Computing with Numbers to Computing with Words – From Manipulation of Measurements to manipulation of Perceptions,* In Proc. Of EUROFUSE -SIC Conf. 1999., Budapest, pp. 1-3.

[15] Zimmermann, H.J., (1991), *Fuzzy Sets, Decision Making and Expert Systems.* Kluwer, Boston, 1991.

[16] Pap, E., *Triangular norms in modelling uncertainly, non-linearity and decision,* in Proceedings of the 2th International Symposium of Hungarian researchers Computational Intelligence, ISBN 963 7154 06 X, pp. 7-18.

[17] Birkhoff, G., (1973), *Lattice theory.* American Mathematical Siciety, Providence

[18] Höhle, U., (1995), *Commutative residual l-monoids,* in [Höhle & Kelement (1995)], pp. 56-106.

[19] Takacs, M., (2003c), *Approximate reasoning with Distance-based Operators and degrees of coincidence,* in Principles of Fuzzy Preference Modelling and Decision Making, edited by Bernard de Baets, János Fodor, Academia Press Gent, 2003., ISBN 90-382-0567-8

[20] G. Hancke, Á. Szeghegyi: *„Nonlinear Control Via TP Model Transformation: The TORA System Example"*, In the Proc.Of the Symposium on Applied Machine Intelligence (SAMI 2004), Herlany, Slovakia, Januar 16-17 2004, pp.333 - 340.