Rational Frenet-Serret Curves and Rotation Minimizing Frames in Spatial Motion Design

Reza Ravani
Department of Computer Engineering
Science & Research Campus
Tehran
Iran
rezaravani@hotmail.com

Ali Meghdari
Department of Mechanical Engineering
Sharif University of Technology
Tehran
Iran
meghdari@sharif.edu

Bahram Ravani
Department of Mechanical and Aeronautical Engineering
University of California-Davis
California-Davis
USA
bravani@ucdavis.edu

***Abstract* - The aim of this paper is to demonstrate that the techniques of Computer Aided Geometric Design such as spatial rational curves and surfaces could be applied to Kinematics, Computer Animation and Robotics. For this purpose we represent a method which utilizes a special class of rational curves called Rational Frenet-Serret (RF) curves for robot trajectory planning. RF curves distinguished by the property that the motion of their Frenet-Serret frame is rational. We describe an algorithm for interpolation of positions by a rational Frenet-Serret motion. Further more we provide an elegant analysis on spatial frames (Frenet-Serret frame and Rotation Minimizing frame) for smooth robot arm motion and investigate their applications in sweep surface modeling.**

***Keywords* - Motion Planning, Rational Frenet-Serret Curves, Rotation Minimizing Frames, Frenet Serret Frames.**

I. INTRODUCTION

In recent years, it has been realized that the methods of Computer Aided Geometric Design (CAGD) provide elegant tools for various tasks in Computer Graphics, Robotics and Kinematics, especially for the design of rigid body motions. One of the first contribution to this research area was the spherical generalization of the de Casteljau algorithm introduced by Shoemake [15] in order to interpolate the orientations of a moving object. More recently Ge and Ravani [3,4] and Park and Ravani [12] presented methods for constructing so-called Bézier motions by generalizing the subdivision algorithm of Bernstein-Bézier curves. Furthermore this concept has been extended to other areas such as spatial kinematics or robotics by introducing so-called rational spline motions. Rational spline motions are characterized by the property that the trajectories of the points of the moving object are rational spline curves, i.e. the trajectories are NURBS (non-uniform rational B-spline) curves [6]. NURBS curves and surfaces became an industrial standard (STEP) for the data exchange between CAD systems. The main advantage of this approach is the data conformity to state of the art CAD systems which allows straightforward data transfer from CAM to CAD systems. So the programming and control of robots could be done by using of CAD data. In general these data specify the desired trajectory of end-effector but not the orientation. In our approach by using Frenet Serret Frame we also specify the desired orientation of end-effector to perform specific tasks such as arc welding, spray painting and scanning of surfaces with robot equipment.

Rational motions can be said to be the direct generalization of rational curves to kinematics. They are defined by the property, that the trajectories of the points of the moving object(s) are (piecewise) rational curves. Therefore we can apply the algorithms of CAGD directly to these curves.

Using this approach the design of a Cartesian motion of an end-effector is usually done by specifying a set of key control configurations which are interpolated or approximated. In interpolation the curve that passes through each control point and in approximation the curve only passes through the end-points. The other control points exert a "pull". So the intermediate points in approximation simply have some influence on the shape of the curve.

The first who applied rational motions to motion design were Ge and Ravani .Their interpolation algorithm is based on rational dual quaternion curves. Another contribution has been given by Johmstone [10] who used normalized rational quaternion curves in order to interpolate orientations of a moving object for animation.

In this paper we discuss a special class of rational motions called Rational Frenet-Serret (RF) and apply this type of rational motions to robot trajectory planning. In application requiring control of the orientation of a rigid body, as its center of mass executes a given path, alignment of body's principal axes with the Frenet Serret frame at each point appear to be the solution. For this purpose we derive a general formula for RF curves and by using this representation the motion of end-effector in 3D space will be achieved.

Our work is also analyzed the Rotation Minimizing Frames (RMF) and Frenet Serret Frames (FSF) which are associated with spatial curves. The RMF finds important applications is animation and motion control, where the orientation of a rigid body must be specified as its center of mass executes a given spatial trajectory. Another application is the construction of swept surfaces defined by the motion of a profile curve along a sweep curve.

This paper is organized as follows: first we are going to introduce basic notations and review some fundamentals in spatial kinematics and the theory of rational motions. In section 3 we review the concept of Frenet Serret Frame and investigate and address a few properties of RF curves and derive a general representation formula for RF curves. In section 4 we give an detailed algorithm for application of RF curves in robot trajectory planning. In next section we study the concept of Rotation Minimizing Frame with computational relations and investigate the sweeping surfaces and application of FSF and RMF for generating these surfaces. Finally in secion 6 we finish with concluded remarks and future works

II. BASIC NOTATIONS

In the sequel we describe the points p in 3-space with the help of homogeneous coordinates $p=(p_0,p_1,p_2,p_3)^T \in R^4 \setminus \{(0,0,0,0)^T\}$. For points not at infinity, i.e. $p_0 \neq 0$, the corresponding inhomogeneous Cartesian coordinates are $\underline{p}=(\underline{p}_1,\underline{p}_2,\underline{p}_3)^T \in R^3$ of the every point p from $\underline{p}_i = p_i/p_0$ where $i=1,2,3$. The homogeneous coordinate vectors p and λp describe the same point for any constant factor $\lambda \neq 0$. By analogously we are going to use homogeneous plane coordinates P^* for the description of planes. A point P lies in the plane P^* if $<P,P^*>=0$, where $<>$ denote the canonical scalar product in R^3 or R^4.

Consider two coinciding coordinate system in Euclidean 3D-space, the fixed coordinate system E^3 ("world coordinate") and the moving coordinate system $\hat{E}^3$. Both coordinate systems are associated with right-handed Cartesian coordinates frames. Frame is an affine extension of a basis: Requires a vector basis plus a point O (the origin): $F=(\vec{v}_1,\vec{v}_2,...,\vec{v}_n,O)$. Points can be described in either coordinate system. We denote the fixed coordinates of a point by p or $\underline{p}$, and the moving coordinates by $\hat{p}$ or $\hat{\underline{p}}$ respectively. In order to convert moving coordinates into fixed coordinates we have to apply the coordinate transformation that maps $\hat{E}^3$ into E^3. Using homogeneous coordinates, this transformation can be represented by a 4×4 matrix

$$M=\left[\begin{array}{ccc|c} 0 & 0 & 0 & v_0 \\ \hline r_{1,1} & r_{1,2} & r_{1,3} & v_1 \\ r_{2,1} & r_{2,2} & r_{2,3} & v_2 \\ r_{3,1} & r_{3,2} & r_{3,3} & v_3 \end{array}\right]=\left[\begin{array}{ccc|c} 0 & 0 & 0 & v_0 \\ \hline & & & v_1 \\ & R & & v_2 \\ & & & v_3 \end{array}\right],\ v_0 \neq 0 \qquad (1)$$

where the 3×3 submatrix $R=(r_{i,j})_{i,j=1,2,3}$ satisfies the orthogonality conditions

$R.R^T = v_0^2 I_3 \quad and \quad \det(R)>0$.

A continuous one-parameter set of positions of $\hat{E}^3$ defines a motion $M=M(t)$ where the parameter t is assumed to be the time. If all element functions of $M=M(t)$ are polynomials of degree n, the motion M is said to be a rational motion of degree n. In this case all points trajectories $p(t)=M(t).\hat{p}$ are rational curves of general degree n. Applying dual quaternion calculus one can prove that every rational motion of degree n can be written

$$M=\left[\begin{array}{ccc|c} 0 & 0 & 0 & \bar{v}_0(d_0^2+d_1^2+d_2^2+d_3^2) \\ \hline & & & v_1 \\ & \bar{v}_0 D & & v_2 \\ & & & v_3 \end{array}\right] \qquad (2)$$

with the dual quaternion $D=(d_0,d)^T$ and

$$D=\begin{pmatrix} d_0^2+d_1^2 -d_2^2-d_3^2 & -2d_0d_3+2d_1d_2 & 2d_0d_2+2d_1d_3 \\ 2d_0d_3+2d_1d_2 & d_0^2-d_1^2+d_2^2-d_3^2 & -2d_0d_1+2d_2d_3 \\ -2d_0d_2+2d_1d_3 & 2d_0d_1+2d_2d_3 & d_0^2-d_1^2-d_2^2+d_3^2 \end{pmatrix} \qquad (3)$$

Here $\bar{v}_0, v=(v_1,v_2,v_3)^T$ and D are polynomials of degree $n-2k, n\, and\, k$, respectively, where $0\le 2k \le n$. Four parameters $d_0,...,d_3$ are Euler's parameters of the rotational part D of the motion [1].

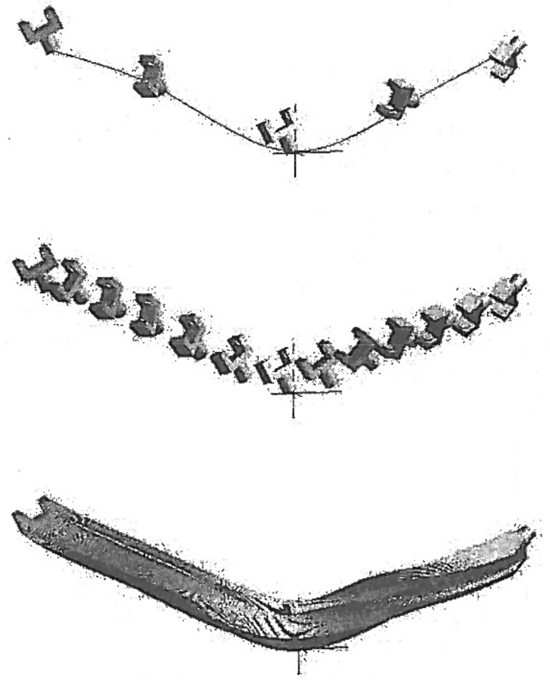

Fig 1: the spational rational b-spline motion of degree 5

The design of rational motions is done most efficiently by computing polynomials $\bar{v}_0, v, D$ that match certain prescribed constraints. The B-Spline representation of M can be obtained by inserting them into (2) [6]. In this case the motion M is expressed by the help of Bernstein polynomials of degree n:

$$M(t)=\sum_{i=0}^{n}B_i^n(t)A_i \qquad (4)$$

where $B_i^n(t)=\binom{n}{i}(1-t)^{n-i}t^i$. Fig 1. shows the spatial rational B-Spline of degree 5 for motion of a robot end-effector.

III. RATIONAL FRENET-SERRET MOTION

At each point of a regular space curve $c(t)$, the Frenet Serret frame defines an orthonormal basis of vectors in R^3 aligned with the local intrinsic curve geometry. The elements of this basis are the curve tangent vector t, normal vector n and binormal vector b. Without loss of generality we may assume that $c(t)$ describers the point path of the origin of the moving coordinate system. Let us further assume that $c(t)$ is a twisted curve without inflection points and with proper parameterization, i.e., $\|\dot{c}\times\ddot{c}\|\neq 0, \|\dot{c}\|\neq 0$ under these assumptions, the motion of the Frenet-serret frame is given by

$$F=\left(\begin{array}{ccc|c} 0 & 0 & 0 & 1 \\ \hline t & n & b & c \end{array}\right) \qquad (5)$$

With

$$t=\frac{\dot{c}}{\|\dot{c}\|},\ n=\frac{(\dot{c}\times\ddot{c})}{\|\dot{c}\times\ddot{c}\|}\times t,\ b=t\times n \qquad (6)$$

In other word we can express the frenet-serret formula by:

$$\begin{bmatrix}\dot{t}\\ \dot{n}\\ \dot{b}\end{bmatrix}=\begin{bmatrix}0 & \kappa & 0\\ -\kappa & 0 & \tau\\ 0 & \tau & 0\end{bmatrix}\begin{bmatrix}t\\ n\\ b\end{bmatrix} \qquad (7)$$

where κ is curvature (see Fig 2) and τ is torsion: a plane curve is completely determined by a single real valued function, the *curvature*, and a space curve is completely determined by two real valued functions, the *curvature* and *torsion*.

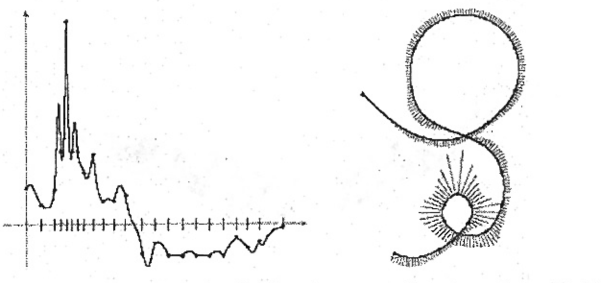

Fig 2: the curvature plot (left) and a porcupine plot of a cubic B-spline curve (right). The endpoints of Bézier segments are indicated on both plots.

In the sequel we will call a motion of type (5) the Frenet-Serret motion of $c(t)$. Taking (5) under consideration it is obvious that the Frenet-Serret motion of a rational curve in general will not be rational. We therefore want to characterize those curves $c(t)$ whose Frenet-Serret motion (5) is rational. Without loss of generality we may assume that $c(t)$ has sufficient differentiability.

Definition1. A curve whose Frenet-Serret motion is rational is called a Rational Frenet-Serret curve (RF curve).

Theorem2. A curve $c(t)$ with nonvanishing curvature $\kappa(t)$ is an RF curve if and only if $c(t)$ is rational and has rational speed $v(t)=\|\dot{c}(t)\|$ and rational curvature $\kappa(t)$.

Proof. The unit vector t, the principal normal vector n and the binormal vector b of $c(t)$ satisfy the Frenet-Serret equations:

$$\begin{aligned} \frac{d}{ds}t &= \kappa n \\ \frac{d}{ds}n &= -\kappa t + \tau b \\ \frac{d}{ds}b &= -\tau n \end{aligned} \qquad (8)$$

Where s denotes the arc length of $c(t)$. rewriting (8) with respect to arbitrary curve parameter t immediately proves the claimed conditions.

A curve with rational speed $v(t)$ is called a rational Pythagorean Hodograph (PH) curve. Such curves has been investigated in a sequence of papers by Farouki and Sakkalis [9], Farouki and Pottmann [8], and Pottmann [14]. Obviously every RF curve is a rational PH curve. On the other hand every planar rational PH curves has rational curvature, which simply yields: A planar curve $c(t)$ is an RF curve if and only if it is a traditional PH curve. In space however, there exist rational PH curves with nonrational curvature. If κ is rational we immediately obtain that the curve defined by the unit tangent vector t of $c(t)$ has to be a rational PH curve as well.

For presenting a general formula for RF curves we consider a rational curve $c(t)$ and its tangent surface Φ which is enveloped by the set of osculating planes of $c(t)$. The curve $c(t)$ is called the line of regression of Φ. Using homogeneous plane coordinates for Φ we obtain the so-called the dual representation $c^*(t)$ of $c(t)$ (see, e.g., [13]), which reads

$$c^*=(-<b,c>,b)^T. \qquad (9)$$

Instead of computing a general representation formula of the point set directly we focus on the dual form (9) of an RF curve.

Theorem3. Let $c(t)$ be an RF curve and $c^*(t)$ is dual representation. Then there exist three pairs $(a,b),(e,f)\ and\ (g,h)$ of polynomials such that

$$C^*=\begin{pmatrix} g(x_0^2+x_1^2+x_2^2)\\ 2hx_0x_1\\ 2hx_0x_2\\ h(x_0^2-x_1^2-x_2^2)\end{pmatrix} \qquad (10)$$

With

$$x_0 = (a^2 + b^2)(ab - ab)f^2$$

$$x_1 = \frac{1}{2}(a^2 - b^2)(\dot{e}f - e\dot{f}) - (a\dot{a} - b\dot{b})ef \qquad (11)$$

$$x_2 = (ab - a\dot{b})ef - ab(\dot{e}f - e\dot{f})$$

proof. According to Farouki and Pottmann[8] each planner rational PH curve with homogenous coordinates $(x_0, x_1, x_2)^T$ has a representation of the type (11). Eq. (10) finally results from $c^* = (g, hb)^T$ by applying the stereographic projection onto the unit sphere.

A parametric representation of the point set $c(t)$ can be obtained by calculating the intersection point of $c^*(t)$ and the first and second derivative plane $\dot{c}^*(t), \ddot{c}^*(t)$. This yield

$$c(t) = c^*(t) \times \dot{c}^*(t) \times \ddot{c}^*(t) \qquad (12)$$

Theorem 3 together with (12) provides a straightforward method for designing RF curves. In Fig 3 you see RF curves for design a Cartesian motion of an end-effector. In Fig 4 RF curves with some inflection points is shown. In this case the orientation of frenet-serret frame does not remain constant based on the curvature changes.

IV. RF CURVES APPLICATIONS

In the following we want to derive an algorithm for the automated design of rational frenet-serret motions that describe the Cartesian space trajectory of a robot's end efector. We assume that the motion is implicitly described by the task level considerations, such that the trajectory of the tool center point is a given curve $c(t)$ and the direction of the z-axis of the hand coordinate system is described by a vector field n(t).

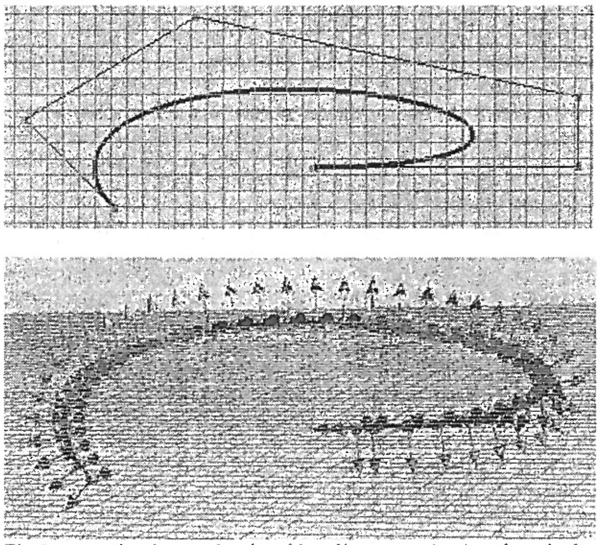

Fig 3: control points and rational b-spline curve (top) and equivalent RF curve with frenet-serret frame

Furthermore we assume that n is normalized, i.e. $\|n(t)\| = 1 \ \forall t$. In arc welding, for example, $c(t)$ would be the seam and n(t) a vector field related to the relative positions of the electrode with respect to the bead of weld. In applications that deal with the scanning of surfaces such as in aircraft inspection on the other hand, $c(t)$ would be a known curve on the surface that has to be scanned and n(t) the corresponding unit normal vector to the surface.

First we observe that $c(t)$ has to be a RF curve if we want to achieve a rational robot trajectory. It is therefore necessary to approximate the tool center point path by a RF curve $c(t)$. Then the orientation of the vectors in frenet-serret frame specify the exact path and direction of end-effector. For this reason we employ the following algorithm:

1. Specifying a set of key control configurations which are interpolated (for end points) and approximated (for other control points or intermediate points). End points have a special concept in robot trajectory planning. It shows the start and goal position.
2. Designing a rational frenet-serret curve based on (12).
3. Achieving a spatial frame (frenet-serret frame) for each point on a trajectory. By means of this frame direction of tool center point of a robot in space could be easily determined. In fact each orthogonal vector (t, n, b) has a special meaning (section 3).

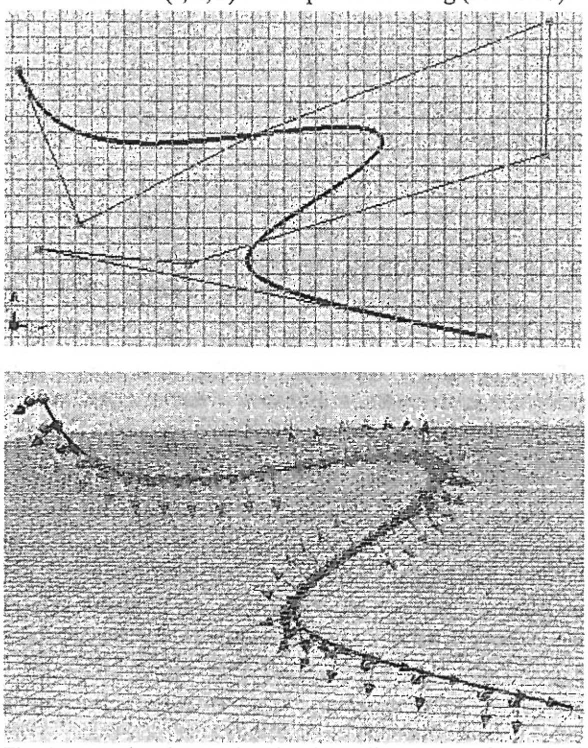

Fig 4: control points and rational b-spline curve with inflection points (top) and equivalent RF curve with frenet-serret frame (bottom)

V. ROTATION MINIMIZING FRAME ON RF CURVES

In application requiring control of the orientation of a rigid body, as its center of mass executes a given path, alignment of the body's principle axes with the Frenet Serret frame at each point may appear to be the obvious solution. However, other useful orthonormal frames (e_1, e_2, e_3) may be defined along a space curve. In most contexts it is natural to choose $e_1 = t$, and (e_2, e_3) are then obtained from (n, b) by a rotation in the normal plane:

$$\begin{bmatrix} e_2 \\ e_3 \end{bmatrix} = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} n \\ b \end{bmatrix} \quad (13)$$

This allow us to remedy indeterminacy of the Frenet Serret frame at inflections, and also provides additional flexibility to adapt the orthonormal frame to the requirements of specific applications. An example is the Rotation Minimizing Frame (RMF) for the construction of swept surfaces, which are defined by the motion of a planar profile curve along a spatial sweep curve. The profile curve remains in the normal plane of the sweep curve, but the variation of its orientation in that plane must be specified. In Computer Aided Geometric Design (CAGD), these frame have firstly beed studied by Klok [11].

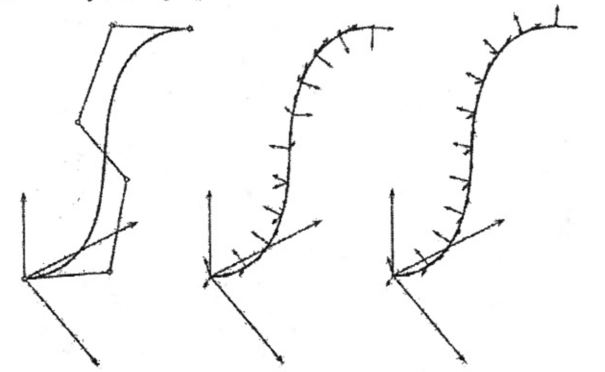

Fig 5: A rational b-spline curve (left), with control polygon. Also shown are the Frenet Serret Frame (center) and Rotation Minimizing Frame (right).

For the purpose of orienting a profile curve along a given sweep curve, the rotation minimizing frame is preferable to the Frenet Serret frame in the following sense (Fig 5) . Considering (7) where dots indicate derivatives with respect to s. This reveals that t changes at instantaneous rate κ in the direction of n. This instantaneous change of n has two components: rate $-\kappa$ in the direction of t and rate τ in the direction of b. Finally b changes at instantaneous rate $-\tau$ in the direction of n. Now changes in the direction of t are unavoidable if we choose a basis with $e_1 = t$. The changes of n in the direction of b, and of b in the direction of n, however, correspond to a rotation of these vectors in the normal plane. A quantitative comparison between FSF and RMF is presented in Fig 6 which shows the instantaneous rates of rotation for both frames.

By a suitable choice for the variation of the angle θ in (13), an orthonormal frame that eliminates this "unnecessary" rotation may be defined. Klok [11] shown that, with $e_1 = t$, the remaining basis vectors must satisfy

$$e_k'(t) = -\frac{c''(t).e_k(t)}{|c'(t)|^2} c'(t), \quad k = 2,3 \quad (14)$$

in order to define such a rotation-minimizing frame. Substituting from (13), one can verify that this amounts to the differential equation

$$\frac{d\theta}{dt} = -|c'|\tau = -|c'|\frac{(c' \times c'').c'''}{|c' \times c''|^2} \quad (15)$$

for the angular function $\theta(t)$ used to obtain (e_2, e_3) from (n, b). Hence, as noted by Guggenheimer [5], this function has the form

$$\theta(t) = \theta_0 - \int_0^t \tau(u)|c'(u)|du \quad (16)$$

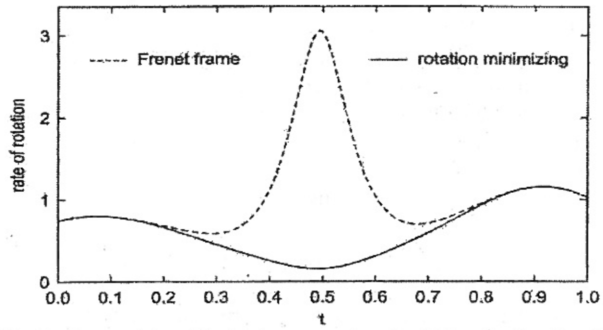

Fig 6: Comparison of instantaneous rates of rotation for the frenet serret frame and rotation minimizing frame.

Unfortunately, the above integral does not admit a closed-form reduction for the polynomial and rational curves employed in computer graphics, computer-aided design, robotics and similar applications. Consequently, a number of schemes have been proposed to approximate the rotation minimizing frame of a given curve, or to approximate a given curve by simpler segments (e.g. circular arcs) with known rotation minimizing frames.

Farouki and Chang [7] have solved integral (16) for PH curves. Since every RF curve is a rational PH curve, we review summary of their works and apply their methods to compute the RMF for RF curves.

For the hodograph $c'(t) = (x'(t), y'(t), z'(t))$ of a polynomial curve to satisfy the pythagorean equation:

$$x'^2(t) + y'^2(t) + z'^2(t) = \sigma^2(t) \quad (17)$$

where $\sigma(t)$ is a polynomial, it is sufficient and necessary that its components be expressible [2] in terms of polynomials $u(t), v(t), p(t), q(t)$ in the forms

$$\begin{aligned} x'(t) &= u^2(t) + v^2(t) - p^2(t) - q^2(t), \\ y'(t) &= 2[u(t)q(t) + v(t)p(t)], \\ z'(t) &= 2[v(t)q(t) - u(t)p(t)], \\ \sigma(t) &= u^2(t) + v^2(t) + p^2(t) + q^2(t) \end{aligned} \quad (18)$$

For PH curves we achieve $|c' \times c''|^2 = \sigma^2 \rho$ where polynomial ρ is defined by

$$\rho = 4[(up' - u'p) + (uq' - u'q) + (vp' - v'p)] + 4[(vq' - v'q)^2 + 2(uv' - u'v)(pq' - p'q)] \quad (19)$$

Based on [7], the solution of integral (16) is

$$\theta(t) = \theta_0 + \int_0 \frac{a(\tau)}{\sigma(\tau)} d\tau + \int_0 \frac{b(\tau)}{\rho(\tau)} d\tau. \quad (20)$$

Where $a(t)$ and $b(t)$ are two polynomial in

$$[c'(t) \times c''(t)].c'''(t) = a(t)\rho(t) + b(t)\sigma(t). \quad (21)$$

Now we want to compare the sweeping surfaces which generated by FSF and RMF. For this reason we study some fundamental concepts of sweeping surfaces.

A sweeping surface $S(u,v)$, where $(u,v) \in [u_0,u_1] \times [v_0,v_1]$ based on a 3D trajectory curve $C(u)$ with unit tangent vector $t(u)$, is defined as $S(u,v) = C(u) + a_1(v)\vec{x}(u) + a_2(v)\vec{y}(u)$ where:

- $F(u) = (\vec{x}(u), \vec{y}(u), \vec{t}(u))$ is a dynamic orthogonal frame along the trajectory;
- $a(v) = (a_1(v), a_2(v))$ is a 2D cross-section curve defined in the abstract XY plane, mapped successively to each plane $t(u)$ normal to the trajectory at $C(u)$;

The moving frame is defined by the tangent vector $t(u)$ of the spine curve, along with unit vectors $\vec{x}(u)$, $\vec{y}(u)$ spanning the normal plane of the spine curve at $C(u)$. The sweeping surface $S(u,v)$ is generated by moving the cross section curve $a(v)$ along the spine curve $C(u)$.

We inspecting two problems that are related to the design of the moving frame $F(u)$ and the associated sweeping surface $S(u,v)$:

- shape: After choosing both the spine and profile curves, the sweeping techniques leaves the designer with one degree of freedom, as it is still possible to rotate the frame $F(u) = (\vec{x}(u), \vec{y}(u), \vec{t}(u))$ around the tangent $\vec{t}$. Clearly, the choice of this rotation has strong influence on the shape of the resulting sweeping surface.
- Rational representations: the peicewise polynomial and rational parametric representations have gained a paramount position as descriptions for curves and surfaces. However, a sweeping surface which is generated by a rational spine curve and a rational cross section curve is generally not rational. In order to apply the Bezier and B-Spline technique to the moving frame, it is desirable that the corresponding moving frame is rational, too.

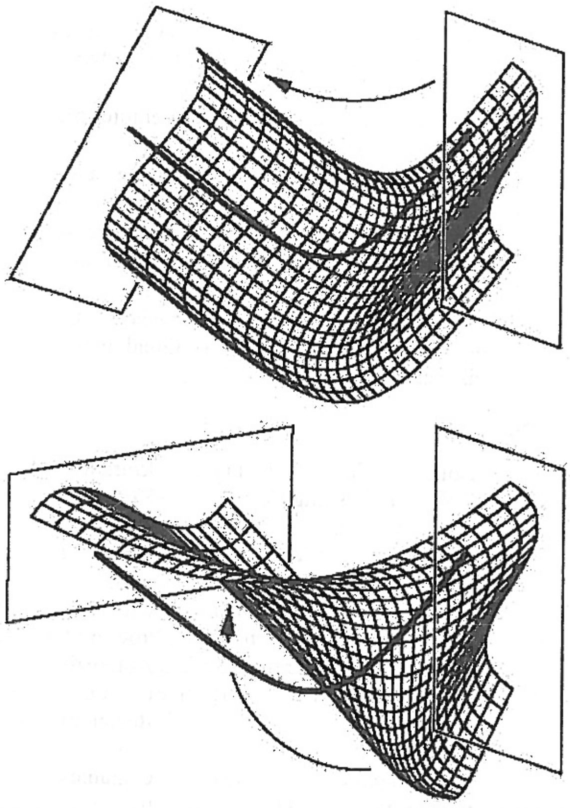

Fig 7: Sweeping surface generated by the Rotation Minimizing Frame (top) and by the Frenet-Serret Frame (bottom).

The shape problem will be handled with the help of RMF that are associated with spatial curves. We are illustrated in Fig 7 the sweeping surface generated by Rotation Minimizing frame and by the Frenet Serret frame of a given spine curve (thick line). In addition to the surfaces, the normal planes of the spine curve at the segment end points have been drawn.

In order to address the problem of finding a rational representation, we use spatial RF curves as spine curves. In our application, these curves are identified as the rational curves wich possess an associated rational frame $F(u)$.

VI. CONCLUSION AND FUTURE WORKS

In this paper we introduced a special type of rational space curves called RF curves. We have proven that a rational curve is an RF curve if and only if speed and curvature are rational. Moreover we provided an application of RF curves in robot trajectory planning. Furthermore the paper showed in various examples that RF curves can be applied successfully to any design algorithm that is based on a spine curve. For many applications, but especially for the design of sweeping surfaces, the Frenet—Serret frame is not the optimal choice since it tends to twist around the curve; therefore, it would be desirable to use a motion that minimizes the angular velocity instead. So the concept of Rotation Minimizing frame and application of these frames in sweeping surfaces presented in detail. We finish this paper by pointing to some topics for further research:

- Generating optimal motions. It is an interesting problem in robotics and NC machining.
- Obstacle avoidance in trajectory planning with RF curves.
- Taking the optimization of RF motions with robot dynamics into account to minimize time or energy functions.
- Advanced CAD/CAM interfaces. By applying the rational motion techniques it is possible to use more sophisticated geometric models.

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